

ON VOLUMES OF COMPLEX HYPERBOLIC ORBIFOLDS

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ABSTRACT. We construct an explicit lower bound for the volume of a complex hyperbolic orbifold that depends only on dimension.

0. INTRODUCTION

A complete rank one locally symmetric space of noncompact type is a quotient of real, complex, or quaternionic hyperbolic space, or the octonionic hyperbolic plane, by a discrete group of isometries, usually denoted by Γ . When the elements of Γ are orientation-preserving, we call the corresponding quotient a *hyperbolic orbifold*. An orbifold is a *manifold* when Γ contains no elements of finite order.

Over each algebra of definition and for each dimension greater than or equal to two, there exists a hyperbolic orbifold of minimum volume. In [2], we produced an explicit lower bound for the volume of a real hyperbolic orbifold, depending only on dimension. See Section 5 of that paper for additional results in the real case. In this article, we turn our attention to the complex setting. Our methods are similar to those of the prequel.

Let $\mathbf{H}_{\mathbb{C}}^n$ denote complex hyperbolic n -space. We identify the group of complex isometries with the Lie group $SU(n, 1)$ and define a *Riemannian submersion* $\pi : SU(n, 1)/\Gamma \rightarrow \mathbf{H}_{\mathbb{C}}^n/\Gamma$. The volume of a complex hyperbolic n -orbifold is thereby described in terms of the volume of the fundamental domain of a lattice in a Lie group. The latter is then bounded from below using results due to Wang [18] and Gunther (see e.g. [5]).

In what follows, we normalize the holomorphic sectional curvature of $\mathbf{H}_{\mathbb{C}}^n$ to be -1 . The sectional curvatures are then pinched between -1 and $-1/4$. Also, dimension will refer to complex dimension, unless stated otherwise.

Theorem 0.1. *The volume of a complex hyperbolic n -orbifold is bounded below by $\mathcal{C}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{C}(n) = \frac{2^{1-n} \pi^{n/2} (n-1)! (n-2)! \cdots 3! 2! 1!}{(9n + 5.25)^{(n^2+2n)/2} \Gamma((n^2 + 2n)/2)} \int_0^{\min[0.1385\sqrt{9n+5.25}, \pi]} \sin^{n^2+2n-1} \rho \, d\rho.$$

The formula of Theorem 0.1 gives the first explicit lower bounds for the volumes of complex hyperbolic orbifolds that depend only on dimension. We get a lower bound of 0.002 for complex 1-orbifolds and 2.918×10^{-9} for complex hyperbolic 2-orbifolds.

Since complex hyperbolic 1-space is isometric to real hyperbolic 2-space, a sharp volume bound of $\pi/21$ for complex hyperbolic 1-orbifolds follows immediately from the classical

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results of Hurwitz [9] and Siegel [16]. For higher complex dimension, explicit volume bounds are known for many categories of orbifolds. See the last section of this article for details.

As in the previous paper [2], we note that volume bounds for hyperbolic orbifolds provide immediate information on the order of the symmetry groups of hyperbolic manifolds. Following Hurwitz's formula for groups acting on surfaces, we have the following corollary.

Corollary 0.2. *Let M be an complex hyperbolic n -manifold. Let H be a group of isometries of M . Then*

$$|H| \leq \frac{\text{Vol}[M]}{\mathcal{C}(n)}.$$

The Chern-Gauss-Bonnet formula (see e.g. [8]) describes volume in terms of Euler characteristic.

$$\text{Vol}(M) = \frac{(-4\pi)^n}{(n+1)!} \chi(M).$$

Hence, we have an alternate version of Corollary 0.2.

Corollary 0.3. *Let M be a finite volume complex hyperbolic n -manifold. Let H be a group of isometries of M . Then*

$$|H| \leq \frac{(-4\pi)^n}{\mathcal{C}(n)(n+1)!} \chi(M).$$

0.1. Outline. The next section gives a definition of complex hyperbolic space and concludes with our preferred symmetric space representation. A comprehensive treatment of complex hyperbolic geometry can be found in [7]. Section 2 provides the background on the geometry of $SU(n, 1)$ that we will use, subsequently. For more details, the reader may consult Chapter 6 of [11] or Sections 1-3 of [2], where we undertook a similar analysis of $SO_o(n, 1)$. In Section 3 we construct an explicit lower bound for the volume of a complex hyperbolic n -orbifold.

1. COMPLEX HYPERBOLIC SPACE

Let $\mathbb{C}^{n,1}$ be a complex vector space of dimension $(n+1)$ equipped with the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z} J \mathbf{w}^* = z_1 \overline{w_1} + z_2 \overline{w_2} + \cdots + z_n \overline{w_n} - z_{n+1} \overline{w_{n+1}}.$$

Here

$$J = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

and $(\cdot)^*$ represents conjugate transpose.

Note that, for all $\mathbf{z} \in \mathbb{C}^{n,1}$ and $\lambda \in \mathbb{C}$, $\langle \mathbf{z}, \mathbf{z} \rangle \in \mathbb{R}$ and $\langle \lambda \mathbf{z}, \lambda \mathbf{z} \rangle = |\lambda|^2 \langle \mathbf{z}, \mathbf{z} \rangle$. Let

$$V_- = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$

and let

$$\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \rightarrow \mathbb{CP}^n$$

be the canonical projection onto complex projective space. *Complex hyperbolic n -space*, $\mathbf{H}_{\mathbb{C}}^n$, is defined to be the space $\mathbb{P}(V_-)$ together with the *Bergman metric*, which is defined by the distance function ρ given by the formula

$$\cosh^2 \left(\frac{\rho(z, w)}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle},$$

where \mathbf{z} and \mathbf{w} are lifts of $z, w \in \mathbf{H}_{\mathbb{C}}^n$.

Denote by $U(n, 1)$ the group of all linear transformation of $\mathbb{C}^{n,1}$ which leave the form $\langle \mathbf{z}, \mathbf{w} \rangle$ invariant. That is,

$$U(n, 1) = \{A \in GL(n+1, \mathbb{C}) | AJA^* = J\},$$

where $GL(n, \mathbb{C})$ is the group of complex nonsingular n -by- n matrices. The *unitary group* is defined and denoted by

$$U(n) = \{A \in GL(n, \mathbb{C}) | AA^* = I\}.$$

The induced action of $U(n, 1)$ on \mathbb{CP}^n preserves $\mathbf{H}_{\mathbb{C}}^n$ and acts by isometries. The stabilizer of the point of $\mathbf{H}_{\mathbb{C}}^n$ with homogeneous coordinates $[0 : \cdots : 0 : 1]$ is

$$U(n) \times U(1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & e^{i\theta} \end{pmatrix} \mid A \in U(n), \theta \in [0, 2\pi) \right\}.$$

Let $A \in U(n)$ and assume $\det A = e^{i\theta}$. We can identify $U(n)$ with $S(U(n) \times U(1))$ by the map

$$A \rightarrow \begin{pmatrix} A & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Hence,

$$\mathbf{H}_{\mathbb{C}}^n = U(n, 1) / (U(n) \times U(1)) = SU(n, 1) / S(U(n) \times U(1)) = SU(n, 1) / U(n).$$

2. THE LIE GROUP $SU(n, 1)$

A Lie group is a group that is also a differentiable manifold, where the group operations are compatible with the smooth structure. A *matrix Lie group* is a closed subgroup of $GL(n, \mathbb{C})$.

Recall that, for a square matrix X ,

$$e^X = I + X + \frac{1}{2}X^2 + \cdots.$$

The Lie algebra of a matrix Lie group G is a vector space, defined as the set of matrices X such that $e^{tX} \in G$, for all real numbers t . The Lie algebra of $GL(n, \mathbb{C})$, denoted by $\mathfrak{gl}(n, \mathbb{C})$, is the set of all $n \times n$ matrices over \mathbb{C} .

The indefinite special unitary group,

$$SU(n, 1) = \{A \in U(n, 1) : \det A = 1\},$$

is a matrix Lie group of real dimension $n^2 + 2n$. The Lie algebra of $SU(n, 1)$ is defined and denoted by

$$\mathfrak{su}(n, 1) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid JX^*J = -X, \text{ trace } X = 0\}.$$

For each n , let $e_{jk} \in \mathfrak{gl}(n+1, \mathbb{C})$ be the matrix with 1 in the jk -position and 0 elsewhere. Furthermore, let $\alpha_{jk} = (e_{jk} - e_{kj})$, $\beta_{jk} = (e_{jk} + e_{kj})$ and $h_j = i(e_{jj} - e_{n+1, n+1})$. A basis for $\mathfrak{su}(n, 1)$ is given below.

Definition 2.1. *The **standard basis** for $\mathfrak{su}(n, 1)$, denoted by \mathfrak{B} , consists of the following set of $n^2 + 2n$ matrices:*

$$\begin{aligned} \alpha_{jk}, \quad 1 \leq j < k \leq n, \quad i\beta_{jk}, \quad 1 \leq j < k \leq n, \\ \beta_{j, n+1}, \quad 1 \leq j \leq n, \quad i\alpha_{j, n+1}, \quad 1 \leq j \leq n, \\ h_j, \quad 1 \leq j \leq n. \end{aligned}$$

The Lie bracket of a matrix Lie algebra is determined by matrix operations.

$$[X, Y] = XY - YX.$$

The following proposition describes the Lie bracket of $\mathfrak{su}(n, 1)$. The proof involves straightforward calculation and is omitted.

Proposition 2.2. *For $1 \leq j < k \leq n, 1 \leq l < m \leq n$,*

$$(2.1) \quad [\alpha_{jk}, \alpha_{lm}] = \delta_{kl}\alpha_{jm} + \delta_{km}\alpha_{lj} + \delta_{jm}\alpha_{kl} + \delta_{lj}\alpha_{mk},$$

$$(2.2) \quad [i\beta_{jk}, i\beta_{lm}] = -(\delta_{kl}\alpha_{jm} + \delta_{km}\alpha_{jl} + \delta_{jm}\alpha_{kl} + \delta_{lj}\alpha_{km}),$$

$$(2.3) \quad [h_j, h_k] = 0,$$

$$(2.4) \quad [\alpha_{jk}, i\beta_{lm}] = i(\delta_{kl}\beta_{jm} + \delta_{km}\beta_{jl} - \delta_{jm}\beta_{kl} - \delta_{lj}\beta_{km}),$$

$$(2.5) \quad [\alpha_{jk}, h_l] = i(\delta_{kl}\beta_{jl} - \delta_{lj}\beta_{kl}),$$

$$(2.6) \quad [h_l, i\beta_{jk}] = \delta_{kl}\alpha_{jl} + \delta_{lj}\alpha_{kl},$$

$$(2.7) \quad [\alpha_{jk}, \beta_{l, n+1}] = \delta_{lk}\beta_{j, n+1} - \delta_{jl}\beta_{k, n+1},$$

$$(2.8) \quad [\alpha_{jk}, i\alpha_{l, n+1}] = i(\delta_{kl}\alpha_{j, n+1} - \delta_{lj}\alpha_{k, n+1}),$$

$$(2.9) \quad [i\beta_{jk}, \beta_{l, n+1}] = i(\delta_{lk}\alpha_{j, n+1} + \delta_{jl}\alpha_{k, n+1}),$$

$$(2.10) \quad [i\beta_{jk}, i\alpha_{l, n+1}] = -(\delta_{lk}\beta_{j, n+1} + \delta_{jl}\beta_{k, n+1}),$$

$$(2.11) \quad [h_j, \beta_{l, n+1}] = i(\delta_{jl}\alpha_{j, n+1} + \alpha_{l, n+1}),$$

$$(2.12) \quad [h_j, i\alpha_{l, n+1}] = -(\delta_{jl}\beta_{j, n+1} + \beta_{l, n+1}),$$

$$(2.13) \quad [\beta_{j, n+1}, \beta_{k, n+1}] = \alpha_{jk},$$

$$(2.14) \quad [i\alpha_{j,n+1}, i\alpha_{k,n+1}] = \alpha_{jk},$$

$$(2.15) \quad [i\alpha_{j,n+1}, \beta_{k,n+1}] = i(\beta_{jk} - 2\delta_{jk}e_{n+1,n+1}).$$

Proposition 2.2 illustrates a *Cartan decomposition* $\mathfrak{su}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$(2.16) \quad \mathfrak{k} = \text{span}\{\alpha_{jk}, i\beta_{jk}, h_j, 1 \leq j < k \leq n\}, \quad \mathfrak{p} = \text{span}\{\beta_{j,n+1}, i\alpha_{j,n+1}, 1 \leq j \leq n\},$$

$$(2.17) \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

2.1. The Canonical Metric of $SU(n, 1)$. For $X \in \mathfrak{su}(n, 1)$, the *adjoint action* of X is the $\mathfrak{su}(n, 1)$ -endomorphism defined by the Lie bracket,

$$\text{ad } X(Y) = [X, Y].$$

The *Killing form* on $\mathfrak{su}(n, 1)$ is a symmetric bilinear form given by

$$B(X, Y) = \text{trace}(\text{ad } X \text{ ad } Y).$$

A positive definite inner product on $\mathfrak{su}(n, 1)$ is then defined by putting

$$\langle X, Y \rangle = \begin{cases} B(X, Y) & \text{for } X, Y \in \mathfrak{p}, \\ -B(X, Y) & \text{for } X, Y \in \mathfrak{k}, \\ 0 & \text{otherwise.} \end{cases}$$

By the standard identification of the tangent space of a Lie group to its Lie algebra we extend $\langle \cdot, \cdot \rangle$ to a left invariant Riemannian metric over $SU(n, 1)$ by left translation. This metric, which we denote by g , will be referred to as the *canonical metric* for $SU(n, 1)$. We denote the induced distance function on $SU(n, 1)$ by ρ .

The following lemma describes the canonical metric for $SU(n, 1)$, with respect to the standard basis.

Lemma 2.3. *For $X, Y \in \mathfrak{B}$,*

$$\langle X, Y \rangle = \begin{cases} 4n + 4 & \text{if } X = Y \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let

$$\begin{aligned} \mathfrak{k}_{jk} &= \text{span}\{\alpha_{jk}, i\beta_{jk}\}, 1 \leq j < k \leq n, \\ \mathfrak{k}_h &= \text{span}\{h_j, j = 1, \dots, n\}, \\ \mathfrak{p}_j &= \text{span}\{\beta_{j,n+1}, i\alpha_{j,n+1}\}, 1 \leq j \leq n. \end{aligned}$$

By (2.5), (2.6), (2.11) and (2.12), for each $h \in \mathfrak{k}_h$, $\text{ad } h(\mathfrak{k}_{jk}) \subset \mathfrak{k}_{jk}$ and $\text{ad } h(\mathfrak{p}_j) \subset \mathfrak{p}_j$. In fact, if $h = \sum_s d_s h_s$, then

$$\begin{aligned} [h, \alpha_{jk}] &= (d_j - d_k)i\beta_{jk}, \\ [h, i\beta_{jk}] &= (d_k - d_j)\alpha_{jk}, \\ [h, \beta_{j,n+1}] &= \left(d_j + \sum_s d_s\right)i\alpha_{j,n+1}, \end{aligned}$$

$$[h, i\alpha_{j,n+1}] = - \left(d_j + \sum_s d_s \right) \beta_{j,n+1}.$$

Therefore

$$\begin{aligned} B(h, h) &= \text{trace}((\text{ad } h)^2) = \sum_{j < k} \text{trace}((\text{ad } h|_{\mathfrak{k}_{jk}})^2) + \sum_j \text{trace}((\text{ad } h|_{\mathfrak{p}_j})^2) \\ &= - \sum_{j,k} (d_j - d_k)^2 - 2 \sum_j \left(d_j + \sum_k d_k \right)^2 \\ &= -(2n+2) \left(\sum_j d_j^2 + \left(\sum_j d_j \right)^2 \right) \\ &= (2n+2)(\text{trace } h)^2. \end{aligned}$$

Since each element of $\mathfrak{su}(n, 1)$ can be diagonalized, for each $X \in \mathfrak{su}(n, 1)$, there is a matrix A such that $AXA^{-1} \in \mathfrak{k}_h$. By the invariance of B and trace, $B(X, X) = 2(n+1) \text{trace}(X^2)$. By polarization,

$$(2.18) \quad B(X, Y) = 2(n+1) \text{trace}(XY), \quad X, Y \in \mathfrak{su}(n, 1).$$

Hence when $X \neq Y \in \mathfrak{B}$, we have $\langle X, X \rangle = 4(n+1)$ and $\langle X, Y \rangle = 0$. \square

Corollary 2.4. *The matrix representation for the canonical metric g of $SU(n, 1)$ is the square $n^2 + 2n$ diagonal matrix*

$$\begin{pmatrix} 4n+4 & & & \\ & 4n+4 & & \\ & & \ddots & \\ & & & 4n+4 \end{pmatrix}.$$

We now make two points relating to the canonical metric that will be significant in our later discussions.

First, we will be interested in the metric on $SU(n, 1)$ that induces holomorphic sectional curvature -1 on the quotient $SU(n, 1)/U(n)$. To this end, we scale the canonical metric by a factor of $\frac{1}{n+1}$. Formally,

Definition 2.5. *Let g be the canonical metric on $SU(n, 1)$. The metric \tilde{g} on $SU(n, 1)$ is defined by*

$$\tilde{g} = \frac{1}{n+1} g.$$

Second, a metric on a Lie algebra \mathfrak{g} induces a norm given by

$$\|X\| = \langle X, X \rangle^{1/2}.$$

Let,

$$N(\text{ad } X) = \sup\{\|\text{ad } X(Y)\| \mid Y \in \mathfrak{g}, \|Y\| = 1\},$$

$$(2.19) \quad C_1 = \sup\{N(\operatorname{ad} X) \mid X \in \mathfrak{p}, \|X\| = 1\} \text{ and } C_2 = \sup\{N(\operatorname{ad} X) \mid X \in \mathfrak{k}, \|X\| = 1\}.$$

The appendix to [18] includes a table of the constants C_1 and C_2 for noncompact and nonexceptional Lie groups. For $SU(n, 1)$, with respect to the scaled canonical metric \tilde{g} , we have

$$(2.20) \quad C_1 = C_2 = 1.$$

2.2. The Sectional Curvatures of $SU(n, 1)$. Let $U, V, W \in \mathfrak{k}$ and $X, Y, Z \in \mathfrak{p}$ denote left invariant vector fields. In [2], we derived the curvature formulas for the canonical metric of a semisimple noncompact Lie group.

Proposition 2.6.

$$(2.21) \quad R(U, V)W = \frac{1}{4}[[V, U], W],$$

$$(2.22) \quad R(X, Y)Z = -\frac{7}{4}[[X, Y], Z],$$

$$(2.23) \quad R(U, X)Y = \frac{1}{4}[[X, U], Y] - \frac{1}{2}[[Y, U], X],$$

$$(2.24) \quad R(X, Y)V = \frac{3}{4}[X, [V, Y]] + \frac{3}{4}[Y, [X, V]].$$

In particular,

$$(2.25) \quad \langle R(U, V)W, X \rangle = 0,$$

$$(2.26) \quad \langle R(X, Y)Z, U \rangle = 0,$$

$$(2.27) \quad \langle R(U, V)V, U \rangle = \frac{1}{4}\| [U, V] \|^2,$$

$$(2.28) \quad \langle R(X, Y)Y, X \rangle = -\frac{7}{4}\| [X, Y] \|^2,$$

$$(2.29) \quad \langle R(U, X)X, U \rangle = \frac{1}{4}\| [U, X] \|^2.$$

Note that these formulas also apply to \tilde{g} , as it is simply a scale of the canonical metric.

Proposition 2.7. *The sectional curvature of $SU(n, 1)$ with respect to the metric \tilde{g} at the planes spanned by standard basis elements is bounded above by $1/4$.*

Proof. Since the basis elements are mutually orthogonal, the sectional curvature at the plane spanned by any distinct elements $X, Y \in \mathfrak{B}$ is given by

$$K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\|X\|^2 \|Y\|^2}.$$

By (2.27), (2.28), (2.29) and Proposition 2.2, the largest sectional curvature spanned by basis directions are the planes spanned by $h_j, i\alpha_{j,n+1}$ or $h_j, \beta_{j,n+1}$. And

$$(2.30) \quad K(h_j, i\alpha_{j,n+1}) = \frac{\frac{1}{4}\| [h_j, i\alpha_{j,n+1}] \|^2}{\|h_j\|^2 \|i\alpha_{j,n+1}\|^2} = \frac{1}{4} \frac{\| -2\beta_{j,n+1} \|^2}{4 \cdot 4} = 1/4.$$

□

Proposition 2.8. *The sectional curvatures of $SU(n, 1)$ with respect to \tilde{g} are bounded above by*

$$\frac{1}{4} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{6}{4} \cdot (2n + 1) + 2 \cdot \frac{3}{4} \cdot (2n + 1) = \frac{36n + 21}{4}.$$

Proof. Again with $U, V \in \mathfrak{k}$ and $X, Y \in \mathfrak{p}$, we have by (2.25) and (2.26)

$$\begin{aligned} \langle R(X + U, Y + V)Y + V, X + U \rangle &= \langle R(X, Y)Y, X \rangle + \langle R(U, V)V, U \rangle + \langle R(U, Y)Y, U \rangle \\ &\quad + \langle R(X, V)V, X \rangle + 2\langle R(X, Y)V, U \rangle + 2\langle R(X, V)Y, U \rangle. \end{aligned}$$

Assume that $\|U + X\| = 1$, $\|V + Y\| = 1$ and $\langle U + X, V + Y \rangle = 0$. Write

$$U = \sum_{j < k} (a_{jk}\alpha_{jk} + b_{jk}i\beta_{jk}) + \sum_j c_j h_j, \quad V = \sum_{j < k} (a'_{jk}\alpha_{jk} + b'_{jk}i\beta_{jk}) + \sum_j c'_j h_j,$$

$$X = \sum_{j=1}^n (e_j i\alpha_{j,n+1} + f_j \beta_{j,n+1}), \quad Y = \sum_{j=1}^n (e'_j i\alpha_{j,n+1} + f'_j \beta_{j,n+1}).$$

Note that

$$\sum_{j < k} |a_{jk}|^2 + |b_{jk}|^2 + c_j^2, \quad \sum_{j < k} |a'_{jk}|^2 + |b'_{jk}|^2 + |c'_j|^2, \quad \sum_{j=1}^n e_j^2 + f_j^2, \quad \sum_{j=1}^n |e'_j|^2 + |f'_j|^2 \leq \frac{1}{4}.$$

By (2.21) and (2.23),

$$R(U, V)V = \frac{1}{4}[[V, U], V] = -\frac{1}{4}\text{ad } V \circ \text{ad } V(U)$$

and

$$R(U, Y)Y = -\frac{1}{4}[[Y, U], Y] = \frac{1}{4}\text{ad } Y \circ \text{ad } Y(U).$$

Therefore, by (2.20),

$$\langle R(U, V)V, U \rangle \leq \frac{1}{4}C_2^2 = \frac{1}{4}$$

and

$$\langle R(U, Y)Y, U \rangle \leq \frac{1}{4}C_1^2 = \frac{1}{4}.$$

By (2.24), we have

$$\langle R(X, Y)V, U \rangle = -\frac{3}{4} (\langle [U, X], [V, Y] \rangle + \langle [V, X], [U, Y] \rangle).$$

From (2.7)–(2.12),

$$\begin{aligned}
\| [U, Y] \|^2 &= \left\| \left[\sum_{j < k} (a_{jk} \alpha_{jk} + b_{jk} i \beta_{jk}) + \sum_j c_j h_j, \sum_{l=1}^n (e'_l i \alpha_{l,n+1} + f'_l \beta_{l,n+1}) \right] \right\|^2 \\
&= \left\| \sum_l \left\{ \left(\sum_j (a_{lj} e'_j + b_{lj} f'_j + c_j f'_l) + c_l f'_l \right) i \alpha_{l,n+1} \right. \right. \\
&\quad \left. \left. + \left(\sum_j (a_{lj} f'_j - b_{lj} e'_j - c_j e'_l) - c_l e'_l \right) \beta_{l,n+1} \right\} \right\|^2 \\
&= 4 \sum_l \left\{ \left(\sum_j (a_{lj} e'_j + b_{lj} f'_j + c_j f'_l) + c_l f'_l \right)^2 + \left(\sum_j (a_{lj} f'_j - b_{lj} e'_j - c_j e'_l) - c_l e'_l \right)^2 \right\} \\
&\leq 4 \sum_l \left(2 \sum_j (a_{lj}^2 + b_{lj}^2 + c_j^2) \cdot \sum_j (|e'_j|^2 + |f'_j|^2 + |f'_l|^2) + 2c_l^2 |f'_l|^2 \right) \\
&\quad + 4 \sum_l \left(2 \sum_j (a_{lj}^2 + b_{lj}^2 + c_j^2) \cdot \sum_j (|f'_j|^2 + |e'_j|^2 + |e'_l|^2) + 2c_l^2 |e'_l|^2 \right) \\
&\leq 8 \sum_l \left(\frac{1}{4} \left[\frac{1}{4} + n |f'_l|^2 \right] + c_l^2 |f'_l|^2 \right) + 8 \sum_l \left(\frac{1}{4} \left[\frac{1}{4} + n |e'_l|^2 \right] + c_l^2 |e'_l|^2 \right) \\
&\leq 2n + 1.
\end{aligned}$$

Here we define $a_{kj} = -a_{jk}$, $b_{kj} = b_{jk}$, $b_{jj} = 0$. Hence

$$\langle R(X, Y)V, U \rangle \leq \frac{6}{4} \cdot (2n + 1).$$

Similarly, by (2.23),

$$\langle R(X, V)Y, U \rangle \leq \frac{3}{4} \cdot (2n + 1).$$

□

3. THE VOLUME OF COMPLEX HYPERBOLIC ORBIFOLDS

This section concludes with a proof of Theorem 0.1. We begin by assembling the required prerequisites. First, a result due to H. C. Wang is used to produce a value such that the fundamental domain of any discrete subgroup Γ of $SU(n, 1)$ contains a metric ball of that radius. Next, a comparison theorem of Gunther is employed in order to bound from below the volume of a ball in $SU(n, 1)$. In the third subsection, a Riemannian submersion from the quotient of $SU(n, 1)$ by Γ onto the complex hyperbolic orbifold defined by Γ is constructed.

3.1. H. C. Wang's Result. Let G be a semisimple Lie group without compact factor. Let C_1 and C_2 be the corresponding constants as defined in (2.19). The number R_G is defined to be the least positive zero of the real-valued function

$$(3.1) \quad F(t) = \exp C_1 t - 1 + 2 \sin C_2 t - \frac{C_1 t}{\exp C_1 t - 1}.$$

The following result (Theorem 5.2 in [18]) gives Wang's quantitative version of the well-known result of Kazhdan-Margulis [12].

Theorem 3.1 (Wang). *Let G be a semisimple Lie group without compact factor, let e be the identity of G , let ρ be the distance function derived from a canonical metric, and let*

$$B_G = \{x \in G : \rho(e, x) \leq R_G\}.$$

Then for any discrete subgroup Γ of G , there exists $g \in G$ such that $B_G \cap g\Gamma g^{-1} = \{e\}$.

In addition, Wang showed that number R_G is less than the injectivity radius of G . Consequently, the volume of the fundamental domain of any discrete subgroup Γ of G is bounded from below by the volume of a ρ -ball of radius $R_G/2$.

As noted in (2.20), for $SU(n, 1)$, with respect to the metric \tilde{g} , $C_1 = C_2 = 1$. Therefore, by (3.1),

$$(3.2) \quad R_{SU(n,1)} = 277/1000.$$

3.2. Gunther's Result. Let $V(d, k, r)$ denote the volume of a ball of radius r in the complete simply connected Riemannian manifold of dimension d with constant curvature k . A proof of the following comparison theorem can be found in [5, Theorem 3.101].

Theorem 3.2 (Gunther). *Let M be a complete Riemannian manifold of dimension d . For $m \in M$, let $B_m(r)$ be a ball which does not meet the cut-locus of m .*

If the sectional curvatures of M are bounded above by a constant b , then

$$\text{Vol}[B_m(r)] \geq V(d, b, r).$$

Proposition 3.3. *Let Γ be a discrete subgroup of $SU(n, 1)$. Then*

$$\text{Vol}[SU(n, 1)/\Gamma] \geq V(d_0, k_0, r_0),$$

where $d_0 = n^2 + 2n$, $k_0 = 9n + 5.25$ and $r_0 = 0.1385$.

Proof. The inequality is immediate from Theorems 3.1 and 3.2. The values of d_0 , k_0 and r_0 follow from Definition 2.1, Proposition 2.8 and (3.2), respectively. \square

3.3. Riemannian Submersions. Let (M, g) and (N, h) be Riemannian manifolds and $q : M \rightarrow N$ a surjective submersion. The map q is said to be a *Riemannian submersion* if

$$g(X, Y) = h(dqX, dqY) \text{ whenever } X, Y \in (\text{Ker } dq)_x^\perp \text{ for some } x \in M.$$

The following elementary results are proved in [2].

Lemma 3.4. *Let G be a semisimple Lie group and let \mathfrak{g} be its Lie algebra, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let K be the maximal compact subgroup of G with Lie algebra \mathfrak{k} . Then, with respect to the canonical metric, K is totally geodesic in G .*

Lemma 3.5. *Let $K \rightarrow M \xrightarrow{q} N$ denote a fiber bundle where q is a Riemannian submersion and K is a compact and totally geodesic submanifold of M . Then for any subset $Z \subset N$,*

$$\text{Vol}[q^{-1}(Z)] = \text{Vol}[Z] \cdot \text{Vol}[K]$$

Let X, Y be orthonormal vector field on N and let \tilde{X}, \tilde{Y} be their horizontal lifts to M . O'Neill's formula (see e.g. [5, Page 127]), relates the sectional curvature of the base space of a Riemannian submersion with that of the total space:

$$(3.3) \quad K_b(X, Y) = K_t(X, Y) + \frac{3}{4} \|[X, Y]^\perp\|^2,$$

where Z^\perp represents the vertical component of Z .

Recall the definitions and notation of Section 2 and consider the quotient map

$$\pi : SU(n, 1) \rightarrow SU(n, 1)/U(n).$$

The restriction of the inner product $\langle X, Y \rangle$, defined on $\mathfrak{su}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$, to $\mathfrak{p} = T_e SU((n, 1)/U(n))$, induces a Riemannian metric on the quotient space. With respect to these metrics, the map π is a Riemannian submersion.

We now show that if $SU(n, 1)/U(n)$ is equipped with the restriction of the scaled canonical metric \tilde{g} , it has constant *holomorphic sectional curvature* -1 . It then follows that π is a Riemannian submersion from $SU(n, 1)$ to $\mathbf{H}_{\mathbb{C}}^n$, complex hyperbolic n -space.

Let $X \in \mathfrak{p}$ represent both a unit vector field on $SU(n, 1)/U(n)$ as well as its horizontal lift. Write $X = \sum_{j=1}^n (a_j i \alpha_{j,n+1} + b_j \beta_{j,n+1})$. Since $\|X\| = 1$, we have $\sum_j (a_j^2 + b_j^2) = \frac{1}{4}$.

From the identification of complex structure

$$\begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \longleftrightarrow \xi \quad \text{for } \xi \in \mathbb{C},$$

$JX = \sum_{k=1}^n (-b_k i \alpha_{k,n+1} + a_k \beta_{k,n+1})$. By (2.28), the holomorphic sectional curvature

$$K_t(X, JX) = \langle R(X, JX)JX, X \rangle = -\frac{7}{4} \|[X, JX]\|^2.$$

By (2.13)-(2.15),

$$\begin{aligned} [X, JX]^\perp &= [X, JX] \\ &= \sum_{j,k} (a_k b_j - a_j b_k) \alpha_{jk} + \sum_{j \neq k} (a_j a_k + b_j b_k) i \beta_{jk} + 2 \sum_j (a_j^2 + b_j^2) h_j \\ &= 2 \left\{ \sum_{j < k} [(a_k b_j - a_j b_k) \alpha_{jk} + (a_j a_k + b_j b_k) i \beta_{jk}] + \sum_j (a_j^2 + b_j^2) h_j \right\}. \end{aligned}$$

Hence

$$\begin{aligned}
K_b(X, JX) &= -\|[X, JX]\|^2 \\
&= -4 \cdot 4 \left\{ \sum_{j < k} [(a_k b_j - a_j b_k)^2 + (a_j a_k + b_j b_k)^2] + \left(\sum_j (a_j^2 + b_j^2)^2 \right) \right\} \\
&= -4 \cdot 4 \left[\sum_j (a_j^2 + b_j^2) \sum_k (a_k^2 + b_k^2) \right] = -1.
\end{aligned}$$

For a discrete group $\Gamma < SU(n, 1)$ and complex hyperbolic orbifold $Q = \mathbf{H}_{\mathbb{C}}^n / \Gamma$, the map π induces another Riemannian submersion

$$\pi : SU(n, 1) / \Gamma \rightarrow Q.$$

The fibers of π on the smooth points of Q are totally geodesic embedded copies of $U(n)$.

3.4. Main Result. We now give a proof of Theorem 0.1, which for convenience is restated below.

Theorem 0.1 *The volume of a complex hyperbolic n -orbifold is bounded below by $\mathcal{C}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{C}(n) = \frac{2^{1-n} \pi^{n/2} (n-1)! (n-2)! \cdots 3! 2! 1!}{(9n + 5.25)^{(n^2+2n)/2} \Gamma((n^2 + 2n)/2)} \int_0^{\min[0.1385\sqrt{9n+5.25}, \pi]} \sin^{n^2+2n-1} \rho \, d\rho.$$

Proof. Let Q be a complex hyperbolic n -orbifold. By the last paragraph of the previous subsection, Proposition 3.3 and Lemma 3.5,

$$V(d_0, k_0, r_0) \leq \text{Vol}[SU(n, 1) / \Gamma] \leq \text{Vol}[\pi^{-1}(Q)] = \text{Vol}[Q] \cdot \text{Vol}[U(n)].$$

The proof follows from the following two observations:

The volumes of the classical compact groups are given explicitly in [6, Page 399]. For the unitary group, the volume with respect to the metric \tilde{g} is

$$\text{Vol}[U(n)] = \frac{2^n \pi^{(n^2+n)/2}}{(n-1)! (n-2)! \cdots 3! 2! 1!}.$$

The complete simply connected Riemannian manifold with constant curvature $k > 0$ is the sphere of radius $k^{-1/2}$. By explicit computation we have

$$V(d, k, r) = \frac{2(\pi/k)^{d/2}}{\Gamma(d/2)} \int_0^{\min[rk^{1/2}, \pi]} \sin^{d-1} \rho \, d\rho.$$

□

4. VOLUME BOUNDS

A complex hyperbolic orbifold is : a *manifold* when Γ does not contain elliptic elements; *cusped* when Γ does contain parabolic elements; *arithmetic* when Γ can be derived by a specific number-theoretic construction (see e.g. [3]). In this section, we give an outline of current results on complex hyperbolic volume.

4.1. Complex Hyperbolic Manifolds. In [8], Hersonsky and Pauline used the Chern-Gauss-Bonnet formula to prove that the smallest volume of a closed, complex hyperbolic 2-manifold is $8\pi^2$. Xie, Wang and Jiang give a lower bound for the volume of complex hyperbolic manifolds, for each dimension [19].

4.2. Cusped Complex Hyperbolic Manifolds. Volume bounds for noncompact complex hyperbolic manifolds in terms of dimension and the number of cusps were given by Hersonsky and Pauline [8] and Parker [13]. These bounds were later improved by Hwang [10] using methods from algebraic geometry. In [13], Parker proved that the smallest volume of a cusped (and so of any) complex hyperbolic 2-manifold is $8\pi^2/3$ and found one such example.

4.3. Complex Hyperbolic Orbifolds. In addition to the results stated above, Parker [13] also proved that the volume of a complex hyperbolic 2-orbifold is bounded below by .25. He identified two orbifolds with volume $\pi^2/27$, and conjectured them to be the cusped complex hyperbolic 2-orbifolds of minimum volume. Extending a result for the real hyperbolic case [1], Fu, Li and Wang obtained a lower bound for the volume of a complex hyperbolic orbifold, depending on dimension and the maximal order of torsion in the orbifold fundamental group [4].

4.4. Arithmetic Complex Hyperbolic Orbifolds. Emery and Stover [3], Stover [17], and Prasad and Yeung [14],[15] have addressed complex hyperbolic volume in the arithmetic setting. In [3], noncompact arithmetic complex hyperbolic n -orbifolds are considered. It is shown that, as n varies, minimum volume is realized in dimension 9. Assisted by work done in [14], [15] on the classification of fake projective planes, Stover [17] proves that the orbifolds considered by Parker in [13] are the smallest volume arithmetic complex hyperbolic 2-orbifolds.

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